Correction model Final Exam Linear Algebra 2
April 3, 2018

1. a) $\text{Span}\left(\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right) = \text{Span}\left(\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right)$. These

are already orthogonal, so we only need normalization: $\left\|\begin{pmatrix}1\\0\\1\\1\end{pmatrix}\right\| = \sqrt{3}$, $\left\|\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\| = 1$. An orthonormal

(8) basis is then $\left\{ \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\}$

b) The closest vector in S to $\begin{pmatrix}a\\b\\a\\b\end{pmatrix}$ is the orthogonal projection

(7) $$P = \left\langle \begin{pmatrix}a\\b\\a\\b\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\1\\1\end{pmatrix} \right\rangle \cdot \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}a\\b\\a\\b\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\rangle \cdot \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$= \frac{2a+b}{3}\begin{pmatrix}1\\0\\1\\1\end{pmatrix} + b\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

2. a) The characteristic polynomial of A is $p_A(s) =$

$$\det\begin{pmatrix} a-s & b \\ c & d-s \end{pmatrix} = (a-s)(d-s) - bc$$

(8) $$= s^2 + (-a-d)s + \cancel{(-bc)}\ ad - bc$$

By Cayley-Hamilton we have $p_A(A) = 0$ so we should choose $\alpha = -a-d$ and $\beta = ad-bc$.

b) For the given matrix A we have

(7) $$p_A(s) = (-1)^4\left(s^4 + ds^3 + cs^2 + bs + a\right)$$

So take $\alpha = d$, $\beta = c$, $\gamma = b$, $\delta = a$

3 a) We first compute $M^TM = \begin{pmatrix} 85 & -30 \\ -30 & 40 \end{pmatrix}$

Then the eigenvalues of M^TM:

$$\det\begin{pmatrix} s-85 & 30 \\ 30 & s-40 \end{pmatrix} = s^2 - 125s + 2500$$

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$$= (s-100)(s-25)$$

$\Rightarrow$ Eigenvalues are $\lambda_1 = 100$, $\lambda_2 = 25$

The singular values are $\sigma_1 = \sqrt{\lambda_1} = 10$ and $\sigma_2 = \sqrt{\lambda_2} = 5$

b) The V matrix is $V = (v_1\ v_2)$ with v_1, v_2 eigenvectors of M^TM corresponding to 100 and 25

$$\begin{pmatrix} 85 & -30 \\ -30 & 40 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 100x \\ 100y \end{pmatrix} \Leftrightarrow \begin{matrix} -15x - 30y = 0 \\ -30x - 60y = 0 \end{matrix}$$

$\Leftrightarrow x + 2y = 0$. This yields: $v_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

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$$\begin{pmatrix} 85 & -30 \\ -30 & 40 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 25x \\ 25y \end{pmatrix} \Leftrightarrow \begin{matrix} 60x - 30y = 0 \\ -30x + 15y = 0 \end{matrix}$$

$\Leftrightarrow -2x + y = 0$. Yields: $v_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\Rightarrow V = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

We want to have $U\Sigma V^T = M$, so must find U s.t. $MV = U\Sigma$, equivalently

$Mv_1 = \sigma_1 u_1$ and $Mv_2 = \sigma_2 u_2$.

Therefore $u_1 = \frac{1}{\sigma_1} M v_1 =$

$$\frac{1}{10}\frac{1}{\sqrt{5}}\begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{10\sqrt{5}}\begin{pmatrix} 10 \\ -20 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} M v_2 =$$

$$\frac{1}{5}\frac{1}{\sqrt{5}}\begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{5\sqrt{5}}\begin{pmatrix} 10 \\ 5 \end{pmatrix} =$$

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow U = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

Singular value decomposition of M:

$$M = \underbrace{\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}}_{U} \cdot \underbrace{\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}}_{\Sigma} \cdot \underbrace{\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}}_{V^T}$$

c) Best rank 1 approximation is

(4) $$\frac{1}{\sqrt{5}}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} =$$

$$\frac{1}{5}\begin{pmatrix} 10 & 0 \\ -20 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 20 & -10 \\ -40 & 20 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 \end{pmatrix}.$$

4. a) A is symmetric and real. Let λ be an eigenvalue of A. Then there exists $x \neq 0$ such that $Ax = \lambda x$. This implies $x^H A x = \lambda x^H x = \lambda \|x\|^2$. I prove now

5 that $x^H A x$ is real:

$$\overline{x^H A x} = (x^H A x)^H = x^H A^T x = x^H A x.$$

Thus $x^H A x$ must be real. We conclude

$$\lambda = \frac{x^H A x}{\|x\|^2} \in \mathbb{R}$$

b) A is symmetric so there exists an orthogonal matrix Q s.t. $Q^T A Q = \Lambda$ with Λ diagonal, eigenvalues on the diagonal:

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

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I will prove that $A \geq 0$ implies $\lambda_i \geq 0$ ($i = 1, \dots n$). Take i arbitrary. Let e_i the ith standard basis vector in $\mathbb{R}^n$ and define $x_i = Q e_i$. Then since $A = Q \Lambda Q^T$ we have

$$0 \leq x_i^T A x_i = e_i^T Q^T Q \Lambda Q^T Q e_i = e_i^T \Lambda e_i = \lambda_i .$$

This shows $\lambda_i \geq 0$.

c) By part b, $A = Q \Lambda Q^T$, Q orthogonal. Take an arbitrary $x \in \mathbb{R}^n$. I will show that $x^T A x \geq 0$. Indeed:

$x^T A x = x^T Q \Lambda Q^T x = (Q^T x)^T \Lambda Q^T x$

Define $y \in \mathbb{R}^n$ by $y = Q^T x$. Then

$(Q^T x)^T \Lambda Q^T x = y^T \Lambda y = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2$

Since $\lambda_i \geq 0$ and $y_i^2 \geq 0$ we find that

$x^T A x = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2 \geq 0$.

5. a) A is $m \times n$, $B = n \times m$, so both AB and BA are well-defined. Let $\lambda \neq 0$. Assume λ an eigenvalue of AB with eigenvector x: $ABx = \lambda x$. Then

(3) $BABx = \lambda Bx$. Note that $Bx \neq 0$ since otherwise λx would be 0, which contradicts $\lambda \neq 0$ and $x \neq 0$. Hence Bx is an eigenvector of BA with eigenvalue λ.

Conversely: same argument

b) $a^T b$ is a (real) number. As 1×1 matrix it has eigenvalue $a^T b \neq 0$.

(2) By part a then $a^T b$ is an eigenvalue of ba^T ($n \times n$ matrix)

c) Write $a^T = (a_1\ a_2 \dots a_n)$, $a_i \in \mathbb{R}$. At least one $a_i \neq 0$ for otherwise $a^T b$ would be 0.

(3) Now $ba^T = b(a_1\ a_2 \dots a_n) = (a_1 b\ a_2 b \dots a_n b)$

Since this matrix consists of columns that are a multiple of b we have $\mathcal{R}(ba^T) = \mathcal{R}(b)$.

d) $\text{rank}(ba^T) = \dim R(ba^T)$
$= \dim R(b)$

3 We know $b \neq 0$ since otherwise we would have $a^T b = 0$.
Hence $\dim R(b) = 1$ so $\text{rank}(ba^T) = 1$

e) By the dimension theorem we have

$$\boxed{\text{rank}(ba^T) + \dim N(ba^T) = n}$$

4 Hence $N(ba^T) = n-1$. There are therefore $n-1$ independent vectors $x_1, \dots x_{n-1}$ s.t. $ba^T \cdot x_i = 0$. Each x_i is an eigenvector with eigenvalue 0 so the geometric multiplicity of eigenvalue 0 is $n-1$.

6. a) Two cases: if $a=b$: Then the eigenvalues are a, a, a, a, algebraic mult. is 4

5 if $a \neq b$: then the eigenvalue are a, a, b, b, both with alg. multiplicity 2

b) $a = b$

c) Assume $a \neq b$. It can be seen that $\dim N(M - aI)$ and $\dim N(M - bI)$ are both equal to 1. Hence the Jordan form is simply $J = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & b \end{pmatrix}$